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Finite N AdS/CFT Correspondence for Abelian and Nonabelian Orbifolds, and Gauge Coupling Unification *

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Abstract

Although the AdS/CFT correspondence is rigorous only for an infinite $N \rightarrow \infty$ stack of D3-branes, it can be fruitfully studied for finite N as a source of gauge structures and choices for chiral fermions and complex scalars which solve the hierarchy problem by a conformal fixed point. We emphasize orbifolds $AdS_5 \times S^5/\Gamma$ where the resulting GFT has $\mathcal{N} = 0$ supersymmetry. The fact that the complex scalars are prescribed by the construction limits the possible spontaneous symmetry breaking. Both abelian and nonabelian Γ are illustrated by simple examples. An accurate $\sin^2\theta$ in electroweak unification can be obtained, suggesting that this approach merits further study.

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I. INTRODUCTION

It has been a challenge over the last fifteen years to make a connection between superstring theory and the real world. The original attempts [1] to identify massless string modes with the familiar degrees of freedom (quarks, gluons, etc.) did not bear fruit so other ideas to make such identification merit exploration.

A very old idea which is basic to string theory is conformal invariance on the world sheet in two dimensions [2–6]. A more recent idea is that conformal invariance in four spacetime dimensions may guide the sought-for connection of superstring, and hence M theory, to observable physics.

Let us briefly outline the basis for AdS/CFT correspondence to set the scene (for a more complete review, see [7]). Consider the type IIB superstring in flat ten-dimensional Minkowski space, and a number N of parallel D3 branes close to each other, filling a $(3+1)$ subspace of the $(9+1)$ spacetime. The system has two types of perturbative excitations: closed and open strings. Closed strings are the excitations in the bulk and open strings end on the D3 branes. At sufficiently low energies ($\ll l_{string}^{-1}$, the string scale), only massless states play a role. The massless closed string states form a type IIB supergravity multiplet: the massless open string states form an $\mathcal{N} = 4$ vector supermultiplet in $(3+1)$ dimensions with interactions described by an $\mathcal{N} = 4$ $SU(N)$ supersymmetric gauge theory.

Next consider the same system but in the background of a D3 brane solution of supergravity:

$$ds^2 = f^{-1/2}(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + f^{1/2}(dr^2 + d\Omega_5^2) \quad (1)$$

with

$$f = 1 + (R/r)^4 \quad \text{and} \quad R = 4\pi g_{string}(\alpha'_{string})^2 N \quad (2)$$

In this case, the energy of an object depends on r : if it is E_r at r then at $r = \infty$ it appears redshifted to $E_\infty = f^{-1/4}E_r$ because of the g_{tt} metric component in Eq.(1). Thus

to an observer at infinity, the $r \rightarrow 0$ excitations in the "throat" appear of lowest energy. There are two types of massless excitations. In the bulk is the IIB supergravity multiplet interacting via supergravity: in the near-horizon region of the throat where $r \ll R$ and $f \sim (R/r)^4$ the geometry is that of $AdS_5 \times S^5$:

$$ds^2 = \left(\frac{r^2}{R^2} \right) (-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + \frac{R^2 dr^2}{r^2} + R^2 d\Omega_5 \quad (3)$$

In both backgrounds there are two decoupled theories in the low-energy limit. One theory in both cases is supergravity in flat space. It is natural to identify the other two theories: $\mathcal{N} = 4$ $SU(N)$ gauge theory in $(3+1)$ spacetime corresponds to type IIB superstring theory on $AdS_5 \times S^5$.

Let us now step back from M theory and address the needs of a theory of the "real world".

In particle phenomenology, the impressive success of the standard theory based on $SU(3) \times SU(2) \times U(1)$ has naturally led to the question of how to extend the theory to higher energies. One is necessarily led by weaknesses and incompleteness in the standard theory. If one extrapolates the standard theory as it stands one finds (approximate) unification of the gauge couplings at $\sim 10^{16}$ GeV. But then there is the *hierarchy* problem of how to explain the occurrence of the tiny dimensionless ratio $\sim 10^{-14}$ of the weak scale to the unification scale. Inclusion of gravity leads to a *super-hierarchy* problem of the ratio of the weak scale to the Planck scale, $\sim 10^{18}$ GeV, an even tinier $\sim 10^{-16}$. Although this is obviously a very important problem about which conformality by itself is not informative, we shall discuss first the hierarchy rather than the super-hierarchy.

There are four well-defined approaches to the hierarchy problem:

- 1. Supersymmetry
- 2. Technicolor.

- 3. Extra dimensions.
- 4. Conformality.

Supersymmetry has the advantage of rendering the hierarchy technically natural, that once the hierarchy is put in to the lagrangian it need not be retuned in perturbation theory. Supersymmetry predicts superpartners of all the known particles and these are predicted to be at or below a TeV scale if supersymmetry is related to the electroweak breaking. Inclusion of such hypothetical states improves the gauge coupling unification. On the negative side, supersymmetry does not explain the origin of the hierarchy.

Technicolor postulates that the Higgs boson is a fermion-antifermion composite bound by a new (technicolor) strong dynamics at or below the TeV scale. This obviates the hierarchy problem. On the minus side, no simple convincing model of technicolor has been found.

Extra dimensions can have a range as large as $1(\text{TeV})^{-1}$ and the gauge coupling unification can happen quite differently than in only four spacetime dimensions. This replaces the hierarchy problem with a different fine-tuning question of why the extra dimension is restricted to a distance corresponding to the weak interaction scale. There is also a potentially serious problem with the proton lifetime.

Conformality is inspired by the AdS/CFT correspondence discussed above superstring duality and assumes that the particle spectrum of the standard model is enriched such that there is a conformal fixed point of the renormalization group at the TeV scale. Above this scale the coupling do not run so the hierarchy is nullified. Instead, the couplings $\alpha_1^{-1}, \alpha_2^{-1}, \alpha_3^{-1}$ run from low energy up to the TeV scale then combine to one energy-independent $\alpha_{\text{conformal}}^{-1}$ (in most cases equal to α_3^{-1}). It is important to realize that the observed difference between $\alpha_1^{-1}, \alpha_2^{-1}$ and $\alpha_3^{-1} = \alpha_{\text{conformal}}^{-1}$ arise in this approach from the group theory associated with embedding $SU(3) \times SU(2) \times U(1)$ in a semi-simple unifying gauge group.

Conformality is the approach followed in this paper. We shall systematically analyse the

compactification of the IIB superstring on $AdS_5 \times S^5/\Gamma$ where Γ is a discrete non-abelian group designed to break all supersymmetries.

Until very recently, the possibility of testing string theory seemed remote, at best. The advent of the AdS/CFT correspondence suggests this point of view may be too pessimistic, since it could lead to $\sim TeV$ evidence for strings. With this thought in mind, we are encouraged to build AdS/CFT models with realistic fermionic structure, and reduce to the standard model below $\sim 1TeV$.

Using AdS/CFT duality, one arrives at a class of gauge field theories of special recent interest. The simplest compactification of a ten-dimensional superstring on a product of an AdS space with a five-dimensional spherical manifold leads to an $\mathcal{N} = 4$ $SU(N)$ supersymmetric gauge theory, well known to be conformally invariant [8]. By replacing the manifold S^5 by an orbifold S^5/Γ one arrives at less supersymmetries corresponding to $\mathcal{N} = 2, 1$ or 0 depending [9] on whether $\Gamma \subset SU(2)$, $SU(3)$, or $\not\subset SU(3)$ respectively, where Γ is in all cases a subgroup of $SU(4) \sim SO(6)$ the isometry of the S^5 manifold.

It was conjectured in [10] that such $SU(N)$ gauge theories are conformal in the $N \rightarrow \infty$ limit. In [11] it was conjectured that at least a subset of the resultant nonsupersymmetric $\mathcal{N} = 0$ theories are conformal even for finite N and that one of this subset provides the right extension of the standard model. Some first steps to check this idea were made in [12]. Model-building based on abelian Γ was studied further in [13,14], arriving in [15] at an $SU(3)^7$ model based on $\Gamma = Z_7$ which has three families of chiral fermions, a correct value for $\sin^2\theta$ and a conformal scale $\sim 10 \text{ TeV}$.

II. ABELIAN ORBIFOLDS

Since, in the context of field-string duality, there has been a shift regarding the relationship of gravity to the standard model of strong and electroweak interactions we shall begin by characterising how gravity fits in, then to suggest more specifically how the standard model fits in to the string framework.

The descriptions of gravity and of the standard model are contained in the string theory. In the string picture in ten spacetime dimensions, or upon compactification to four dimensions, there is a massless spin-two graviton but the standard model is not manifest in the way we shall consider it. In the conformal field theory extension of the standard model, gravity is strikingly absent. The field-string duality does not imply that the standard model already contains gravity and, in fact, it does not.

In the field theory description [11–14] used in this article, one will simply ignore the massless spin-two graviton. Indeed, since we are using the field theory description only below the conformal scale of $\sim 1\text{TeV}$ (or, as suggested later in this paper, 10TeV) and forgoing any requirement of grand unification, the hierarchy between the weak scale and theory-generated scales like M_{GUT} or M_{PLANCK} is resolved. Moreover, seeking the graviton in the field theory description is possibly resolvable by going to a higher dimension and restricting the range of the higher dimension. Here we are looking only at the strong and weak interactions at accessible energies below, say, 10TeV .

Of course, if we ask questions in a different regime, for example about the scattering of particles with center-of-mass energy of the order M_{PLANCK} , then the graviton will become crucial [16,17] and a string, rather than a field, description will be the viable one.

It is important to distinguish between the holographic description of the five-dimensional gravity in $(AdS)_5$ made by the four-dimensional CFT and the origin of the four-dimensional graviton. The latter could be described holographically only by a lower three-dimensional field theory which is not relevant to the real world. Therefore the graviton of our world

can only arise by *compactification* of a higher dimensional graviton. Introduction of gravity must break conformal invariance and it is an interesting question whether this breaking is related to the mass and symmetry-breaking scales in the low-energy theory. That is all I will say about gravity in the present paper; the remainder is on the standard model and its embedding in a CFT.

An alternative to conformality, grand unification with supersymmetry, leads to an impressively accurate gauge coupling unification [18,19]. In particular it predicts an electroweak mixing angle at the Z-pole, $\sin^2\theta = 0.231$. This result may, however, be fortuitous, but rather than abandon gauge coupling unification, we can rederive $\sin^2\theta = 0.231$ in a different way by embedding the electroweak $SU(2) \times U(1)$ in $SU(N) \times SU(N) \times SU(N)$ to find $\sin^2\theta = 3/13 \simeq 0.231$ [14]. This will be a common feature of the models in this paper.

Actually, it may be premature to dismiss as accidental the success of grand unification with $\sin^2\theta$ since the principal topic here (AdS/CFT) teaches us that quite different theoretical descriptions can be "dual" and the same may eventually be understood for conformality and grand unification. For example, conformality is compatible with supersymmetry at low-energy.

The conformal theories will be finite without quadratic or logarithmic divergences. This requires appropriate equal numbers of fermions and bosons which can cancel in loops and which occur without the necessity of space-time supersymmetry. As we shall see in one example, it is possible to combine spacetime supersymmetry with conformality but the latter is the driving principle and the former is merely an option: additional fermions and scalars are predicted by conformality in the TeV range [13,14], but in general these particles are different and distinguishable from supersymmetric partners. The boson-fermion cancellation is essential for the cancellation of infinities, and will play a central role in the calculation of the cosmological constant (not discussed here). In the field picture, the cosmological constant measures the vacuum energy density.

Here we shall focus on abelian orbifolds characterised by the discrete group Z_p . Non-abelian orbifolds will be discussed in the next section.

The steps in building a model for the abelian case (parallel steps hold for non-abelian orbifolds) are:

- (1) Choose the discrete group Γ . Here we are considering only $\Gamma = Z_p$. We define $\alpha = \exp(2\pi i/p)$.
- (2) Choose the embedding of $\Gamma \subset SU(4)$ by assigning $\mathbf{4} = (\alpha^{A_1}, \alpha^{A_2}, \alpha^{A_3}, \alpha^{A_4})$ such that $\sum_{q=1}^{q=4} A_q = 0(\text{mod } p)$. To break $\mathcal{N} = 4$ supersymmetry to $\mathcal{N} = 0$ (or $\mathcal{N} = 1$) requires that none (or one) of the A_q is equal to zero (mod p).
- (3) For chiral fermions one requires that $\mathbf{4} \not\equiv \mathbf{4}^*$ for the embedding of Γ in $SU(4)$.

The chiral fermions are in the bifundamental representations of $SU(N)^p$

$$\sum_{i=1}^{i=p} \sum_{q=1}^{q=4} (N_i, \bar{N}_{i+A_q}) \quad (4)$$

If $A_q = 0$ we interpret (N_i, \bar{N}_i) as a singlet plus an adjoint of $SU(N)_i$.

- (4) The $\mathbf{6}$ of $SU(4)$ is real $\mathbf{6} = (a_1, a_2, a_3, -a_1, -a_2, -a_3)$ with $a_1 = A_1 + A_2$, $a_2 = A_2 + A_3$, $a_3 = A_3 + A_1$ (recall that all components are defined modulo p). The complex scalars are in the bifundamentals

$$\sum_{i=1}^{i=p} \sum_{j=1}^{j=3} (N_i, \bar{N}_{i \pm a_j}) \quad (5)$$

The condition in terms of a_j for $\mathcal{N} = 0$ is $\sum_{j=1}^{j=3} (\pm a_j) \neq 0(\text{mod } p)$ [11].

- (5) Choose the N of $\otimes_i SU(N d_i)$ (where the d_i are the dimensions of the representations of Γ). For the abelian case where $d_i \equiv 1$, it is natural to choose $N = 3$ the largest $SU(N)$ of the standard model (SM) gauge group. For a non-abelian Γ with $d_i \neq 1$ the choice $N = 2$ would be indicated.
- (6) The p quiver nodes are identified as color (C), weak isospin (W), or a third $SU(3)$ (H). This specifies the embedding of the gauge group $SU(3)_C \times SU(3)_W \times SU(3)_H \subset \otimes SU(N)^p$.

This quiver node identification is guided by (7), (8) and (9) below.

- (7) The quiver node identification is required to give three chiral families under Eq.(4). It is sufficient to make three of the $(C + A_q)$ to be W and the fourth H, given that there is only one C quiver node, so that there are three $(3, \bar{3}, 1)$. Provided that $(\bar{3}, 3, 1)$ is avoided by the $(C - A_q)$ being H, the remainder of the three family trinification will be automatic by chiral anomaly cancellation. Actually, a sufficient condition for three families has been given; it is necessary only that the difference between the number of $(3 + A_q)$ nodes and the number of $(3 - A_q)$ nodes which are W be equal to three.
- (8) The complex scalars of Eq. (5) must be sufficient for their vacuum expectation values (VEVs) to spontaneously break $SU(3)^p \longrightarrow SU(3)_C \times SU(3)_W \times SU(3)_H \longrightarrow SU(3)_C \times SU(2)_W \times U(1)_Y \longrightarrow SU(3)_C \times U(1)_Q$.

Note that, unlike grand unified theories (GUTs) with or without supersymmetry, the Higgs scalars are here prescribed by the conformality condition. This is more satisfactory because it implies that the Higgs sector cannot be chosen arbitrarily.

- (9) Gauge coupling unification should apply at least to the electroweak mixing angle $\sin^2\theta = g_Y^2/(g_2^2 + g_Y^2) \simeq 0.231$. For trinification $Y = 3^{-1/2}(-\lambda_{8W} + 2\lambda_{8H})$ so that $(3/5)^{1/2}Y$ is correctly normalized. If we make $g_Y^2 = (3/5)g_1^2$ and $g_2^2 = 2g_1^2$ then $\sin^2\theta = 3/13 \simeq 0.231$ with sufficient accuracy.

In the remainder of this section we answer all these steps for the choice $\Gamma = Z_p$ for successive $p = 2, 3, \dots$ up to $p = 7$, then add some concluding remarks.

• $p = 2$

In this case $\alpha = -1$ and therefore one cannot construct any complex $\mathbf{4}$ of $SU(4)$ with $\mathbf{4} \not\equiv \mathbf{4}^*$. Chiral fermions are therefore impossible.

- **p = 3**

The only possibilities are $A_q = (1, 1, 1, 0)$ or $A_q = (1, 1, -1, -1)$. The latter is real and leads to no chiral fermions. The former leaves $\mathcal{N} = 1$ supersymmetry and is a simple three-family model [9] by the quiver node identification C - W - H. The scalars $a_j = (1, 1, 1)$ are sufficient to spontaneously break to the SM. Gauge coupling unification is, however, missing since $\sin^2\theta = 3/8$, in bad disagreement with experiment.

- **p = 4**

The only complex $\mathcal{N} = 0$ choice is $A_q = (1, 1, 1, 1)$. But then $a_j = (2, 2, 2)$ and any quiver node identification such as C - W - H - H has 4 families and the scalars are insufficient to break spontaneously the symmetry to the SM gauge group.

- **p = 5**

The two inequivalent complex choices are $A_q = (1, 1, 1, 2)$ and $A_q = (1, 3, 3, 3)$. By drawing the quiver, however, and using the rules for three chiral families given in (7) above, one finds that the node identification and the prescription of the scalars as $a_j = (2, 2, 2)$ and $a_j = (1, 1, 1)$ respectively does not permit spontaneous breaking to the standard model.

- **p = 6**

Here we can discuss three inequivalent complex possibilities as follows:

(6A) $A_q = (1, 1, 1, 3)$ which implies $a_j = (2, 2, 2)$.

Requiring three families means a node identification C - W - X - H - X - H where X is either W or H. But whatever we choose for the X the scalar representations are insufficient to break $SU(3)^6$ in the desired fashion down to the standard theory. This illustrates the difficulty of model building when the scalars are not in arbitrary representations.

(6B) $A_q = (1, 1, 2, 2)$ which implies $a_j = (2, 3, 3)$.

Here the family number can be only zero, two or four as can be seen by inspection of the A_q and the related quiver diagram. So (6B) is of no phenomenological interest.

(6C) $A_q = (1, 3, 4, 4)$ which implies $a_j = (1, 1, 4)$.

Requiring three families needs a quiver node identification which is of the form *either* C - W - H - H - W - H *or* C - H - H - W - W - H. The scalar representations implied by $a_j = (1, 1, 4)$ are, however, easily seen to be insufficient to do the required spontaneous symmetry breaking (S.S.B.) for both of these identifications.

• **p = 7**

Having been stymied mainly by the rigidity of the scalar representation for all $p \leq 6$, for $p = 7$ there are the first cases which work. Six inequivalent complex embeddings of $Z_7 \subset SU(4)$ require consideration.

(7A) $A_q = (1, 1, 1, 4) \implies a_j = (2, 2, 2)$

For the required nodes C - W - X - H - H - X - H the scalars are *insufficient* for S.S.B.

(7B) $A_q = (1, 1, 2, 3) \implies a_j = (2, 3, 3)$

The node identification C - W - H - W - H - H - H leads to a *successful* model.

(7C) $A_q = (1, 2, 2, 2) \implies a_j = (3, 3, 3)$

Choosing C - H - W - X - X - H - H to derive three families, the scalars *fail* in S.S.B.

(7D) $A_q = (1, 3, 5, 5) \implies a_j = (1, 1, 3)$

The node choice C - W - H - H - H - W - H leads to a *successful* model. This is Model A of [14].

(7E) $A_q = (1, 4, 4, 5) \implies a_j = (1, 2, 2)$

The nodes C - H - H - H - W - W - H are *successful*.

$$(7F) \ A_q = (2, 4, 4, 4) \implies a_j = (1, 1, 1)$$

Scalars *insufficient* for S.S.B.

The three successful models (7B), (7D) and (7E) lead to an $\alpha_3(M) \simeq 0.07$. Since $\alpha_3(1\text{TeV}) \geq 0.10$ this suggest a conformal scale $M \simeq 10 \text{ TeV}$ [14]. The above models have less generators than an $E(6)$ GUT and thus $SU(3)^7$ merits further study. It is possible, and under investigation, that non-abelian orbifolds will lead to a simpler model.

For such field theories it is important to establish the existence of a fixed manifold with respect to the renormalization group. It could be a fixed line but more likely, in the $\mathcal{N} = 0$ case, a fixed point. It is known that in the $N \longrightarrow \infty$ limit the theories become conformal, but although this 't Hooft limit [20] is where the field-string duality is derived we know that finiteness survives to finite N in the $\mathcal{N} = 4$ case [8] and this makes it plausible that at least a conformal point occurs also for the $\mathcal{N} = 0$ theories with $N = 3$ derived above.

The conformal structure cannot by itself predict all the dimensionless ratios of the standard model such as mass ratios and mixing angles because these receive contributions, in general, from soft breaking of conformality. With a specific assumption about the pattern of conformal symmetry breaking, however, more work should lead to definite predictions for such quantities.

III. NONABELIAN ORBIFOLDS

Abelian orbifolds lead us to consider the finite N value $N = 3$ guided by trinification $SU(3)^3$ and the fact that all representations of abelian groups Z_p are one dimensional.

A nonabelian orbifold can allow the consideration of finite $N = 2$ since for a Γ with doublet and singlet representations can lead to a generalization of a left-right structure of the type: $SU(4) \times SU(2) \times SU(2)$.

First we remind the reader of available nonabelian Γ of low order $g \leq 31$.

Of the nonabelian finite groups, the best known are perhaps the permutation groups S_N (with $N \geq 3$) of order $N!$ The smallest non-abelian finite group is S_3 ($\equiv D_3$), the symmetry of an equilateral triangle with respect to all rotations in a three dimensional sense. This group initiates two infinite series, the S_N and the D_N . Both have elementary geometrical significance since the symmetric permutation group S_N is the symmetry of the N-plex in N dimensions while the dihedral group D_N is the symmetry of the planar N-agon in 3 dimensions. As a family symmetry, the S_N series becomes uninteresting rapidly as the order and the dimensions of the representations increase. Only S_3 and S_4 are of any interest as symmetries associated with the particle spectrum [21]; also, the order (number of elements) of the S_N groups grow factorially with N . The order of the dihedral groups increases only linearly with N and their irreducible representations are all one- and two- dimensional. This is reminiscent of the representations of the electroweak $SU(2)_L$ used in Nature.

Each D_N is a subgroup of $O(3)$ and has a counterpart double dihedral group Q_{2N} , of order $4N$, which is a subgroup of the double covering $SU(2)$ of $O(3)$.

With only the use of D_N , Q_{2N} , S_N and the tetrahedral group T (of order 12, the even permutations subgroup of S_4) we find 32 of the 45 nonabelian groups up to order 31, either as simple groups or as products of simple nonabelian groups with abelian groups: (Note that $D_6 \simeq Z_2 \times D_3$, $D_{10} \simeq Z_2 \times D_5$ and $D_{14} \simeq Z_2 \times D_7$)

g	
6	$D_3 \equiv S_3$
8	$D_4, Q = Q_4$
10	D_5
12	D_6, Q_6, T
14	D_7
16	$D_8, Q_8, Z_2 \times D_4, Z_2 \times Q$
18	$D_9, Z_3 \times D_3$
20	D_{10}, Q_{10}
22	D_{11}
24	$D_{12}, Q_{12}, Z_2 \times D_6, Z_2 \times Q_6, Z_2 \times T,$
	$Z_3 \times D_4, Z_3 \times Q, Z_4 \times D_3, S_4$
26	D_{13}
28	D_{14}, Q_{14}
30	$D_{15}, D_5 \times Z_3, D_3 \times Z_5$

There remain thirteen others formed by twisted products of abelian factors. Only certain such twistings are permissable, namely (completing all $g \leq 31$).

g	
16	$Z_2 \tilde{\times} Z_8$ (two, excluding D_8), $Z_4 \tilde{\times} Z_4, Z_2 \tilde{\times} (Z_2 \times Z_4)$ (two)
18	$Z_2 \tilde{\times} (Z_3 \times Z_3)$
20	$Z_4 \tilde{\times} Z_5$
21	$Z_3 \tilde{\times} Z_7$
24	$Z_3 \tilde{\times} Q, Z_3 \tilde{\times} Z_8, Z_3 \tilde{\times} D_4$
27	$Z_9 \tilde{\times} Z_3, Z_3 \tilde{\times} (Z_3 \times Z_3)$

It can be shown that these thirteen exhaust the classification of *all* inequivalent finite groups up to order thirty-one [22,23].

Of the 45 nonabelian groups, the dihedrals (D_N) and double dihedrals (Q_{2N}), of order $2N$ and $4N$ respectively, form the simplest sequences. In particular, they fall into subgroups of $O(3)$ and $SU(2)$ respectively, the two simplest nonabelian continuous groups.

For the D_N and Q_{2N} , the multiplication tables, as derivable from the character tables, are, in general, simple to express. D_N , for odd N , has two singlet representations $1, 1'$ and $m = (N - 1)/2$ doublets $2_{(j)}$ ($1 \leq j \leq m$). The multiplication rules are:

$$1' \times 1' = 1; \quad 1' \times 2_{(j)} = 2_{(j)} \quad (6)$$

$$2_{(i)} \times 2_{(j)} = \delta_{ij}(1 + 1') + 2_{(\min[i+j, N-i-j])} + (1 - \delta_{ij})2_{(|i-j|)} \quad (7)$$

For even N , D_N has four singlets $1, 1', 1'', 1'''$ and $(m - 1)$ doublets $2_{(j)}$ ($1 \leq j \leq m - 1$) where $m = N/2$ with multiplication rules:

$$1' \times 1' = 1'' \times 1'' = 1''' \times 1''' = 1 \quad (8)$$

$$1' \times 1'' = 1'''; 1'' \times 1''' = 1'; 1''' \times 1' = 1'' \quad (9)$$

$$1' \times 2_{(j)} = 2_{(j)} \quad (10)$$

$$1'' \times 2_{(j)} = 1''' \times 2_{(j)} = 2_{(m-j)} \quad (11)$$

$$2_{(j)} \times 2_{(k)} = 2_{|j-k|} + 2_{(\min[j+k, N-j-k])} \quad (12)$$

(if $k \neq j, (m - j)$)

$$2_{(j)} \times 2_{(j)} = 2_{(\min[2j, N-2j])} + 1 + 1' \quad (13)$$

(if $j \neq m/2$)

$$2_{(j)} \times 2_{(m-j)} = 2_{|m-2j|} + 1'' + 1''' \quad (14)$$

(if $j \neq m/2$)

$$2_{m/2} \times 2_{m/2} = 1 + 1' + 1'' + 1''' \quad (15)$$

This last is possible only if m is even and hence if N is divisible by *four*.

For Q_{2N} , there are four singlets $1, 1', 1'', 1'''$ and $(N - 1)$ doublets $2_{(j)}$ ($1 \leq j \leq (N - 1)$).

The singlets have the multiplication rules:

$$1 \times 1 = 1' \times 1' = 1 \quad (16)$$

$$1'' \times 1'' = 1''' \times 1''' = 1' \quad (17)$$

$$1' \times 1'' = 1''' ; 1''' \times 1' = 1'' \quad (18)$$

for $N = (2k + 1)$ but are identical to those for D_N when $N = 2k$.

The products involving the $2_{(j)}$ are identical to those given for D_N (N even) above.

This completes the multiplication rules for 19 of the 45 groups. The complete multiplication tables for all the nonabelian groups with order $g \leq 31$ are provided in Appendix A of [24].

Mathematical Theorem: A Pseudoreal 4 of $SU(4)$ Cannot Yield Chiral Fermions.

In [13] it was proved that if the embedding in $SU(4)$ is such that the $\mathbf{4}$ is real, the resultant fermions are always non-chiral. It was implied there that the converse holds, that if $\mathbf{4}$ is complex, $\mathbf{4} = \mathbf{4}^*$, then the resulting fermions are necessarily chiral. Actually for $\Gamma \subset SU(2)$ one encounters the intermediate possibility that the $\mathbf{4}$ is *pseudoreal*. In the present section we shall show that if $\mathbf{4}$ is pseudoreal then the resultant fermions are necessarily non-chiral. The converse now holds: if the $\mathbf{4}$ is neither real nor pseudoreal then the resultant fermions are chiral.

For $\Gamma \subset SU(2)$ it is important that the embedding be consistent with the chain $\Gamma \subset SU(2) \subset SU(4)$, otherwise the embedding is not a consistent one. One way to see the inconsistency is to check the reality of the $\mathbf{6} = (\mathbf{4} \otimes \mathbf{4})_{\text{antisymmetric}}$. If $\mathbf{6} \neq \mathbf{6}^*$ then the embedding is clearly improper. To avoid this inconsistency it is sufficient to include in the $\mathbf{4}$ of $SU(4)$ only complete irreducible representations of $SU(2)$.

An explicit example will best illustrate this propriety constraint on embeddings. Let us consider $\Gamma = Q_6$, the dicyclic group of order $g = 12$. This group has six inequivalent irreducible representations: $1, 1', 1'', 1''', 2_1, 2_2$. The $1, 1', 2_1$ are real. The $1''$ and $1'''$ are a complex conjugate pair, The 2_2 is pseudoreal. To embed $\Gamma = Q_6 \subset SU(4)$ we must choose from the special combinations which are complete irreducible representations of $SU(2)$ namely $1, 2 = 2_2, 3 = 1' + 2_1$ and $4 = 1'' + 1''' + 2_2$. In this way the embedding either makes the $\mathbf{4}$ of $SU(4)$ real *e.g.* $4 = 1 + 1' + 2_1$ and the theorem of [13] applies, and non-chirality results, or the $\mathbf{4}$ is pseudoreal *e.g.* $4 = 2_2 + 2_2$. In this case one can check that the embedding is consistent because $(\mathbf{4} \otimes \mathbf{4})_{\text{antisymmetric}}$ is real. But it is equally easy to check that the product of this pseudoreal $\mathbf{4}$ with the complete set of irreducible representations of Q_6 is again real and that the resultant fermions are non-chiral.

The lesson is:

To obtain chiral fermions from compactification on $AdS_5 \times S_5/\Gamma$, the embedding of Γ in $SU(4)$ must be such that the $\mathbf{4}$ of $SU(4)$ is neither real nor pseudoreal.

Now we are ready for a successful example [23,24]

Group 24/7; also designated $D_4 \times Z_3$

This has twelve singlets $1_1\alpha^i, 1_2\alpha^i, 1_3\alpha^i, 1_4\alpha^i$ ($i = 0 - 2$) and three doublets $2\alpha^i$ ($i = 0 - 2$); here $\alpha = \exp(i\pi/3)$. The embedding $\mathbf{4} = (1_1\alpha, 1_2, 2\alpha)$ was studied in detail in a previous

article [23] where it was shown how it can lead to precisely three chiral families in the standard model. For completeness we include the table [24] for the chiral fermions (it was presented in a different equivalent way in [23]):

	1_1	1_2	1_3	1_4	2	$1_1\alpha$	$1_2\alpha$	$1_3\alpha$	$1_4\alpha$	2α	$1_1\alpha^2$	$1_2\alpha^2$	$1_3\alpha^2$	$1_4\alpha^2$	$2\alpha^2$
1_1		\times				\times				\times					
1_2	\times						\times			\times					
1_3				\times				\times		\times					
1_4			\times						\times	\times					
2					\times	\times	\times	\times	\times	\times					
$1_1\alpha$							\times				\times				\times
$1_2\alpha$						\times						\times			\times
$1_3\alpha$									\times				\times		\times
$1_4\alpha$								\times						\times	\times
2α										\times	\times	\times	\times	\times	\times
$1_1\alpha^2$	\times				\times							\times			
$1_2\alpha^2$		\times			\times						\times				
$1_3\alpha^2$			\times		\times									\times	
$1_4\alpha^2$				\times	\times								\times		
$2\alpha^2$	\times	\times	\times	\times	\times										\times

By identifying $SU(4)$ with the diagonal subgroup of $SU(4)_{2,3}$, breaking $SU(4)_1$ to $SU(2)'_L \times SU(2)'_R$, then identifying $SU(2)_L$ with the diagonal subgroup of $SU(2)_{6,7,8}$ and $SU(2)'_L$ and $SU(2)_R$ with the diagonal subgroup of $SU(2)_{10,11,12}$ and $SU(2)'_R$ then leads to a three-family model.

This model is especially interesting because, uniquely among the large number of models

examined in this study, the prescribed scalars are sufficient to break the gauge symmetry to that of the standard model with three chiral families.

IV. GAUGE COUPLING UNIFICATION

Most of the research beyond the standard model [25] is motivated by the hierarchy problem and uses the two assumptions of grand unification and low-energy ($\sim TeV$) supersymmetry. This is, in turn, driven largely by the successful prediction of one number, $\sin^2\theta$ of the electroweak mixing angle θ . It is proposed to replace the two assumptions of grand unification and low-energy supersymmetry by one assumption, conformality. It therefore is important to show that $\sin^2\theta$ can be derived from conformality alone; that is the principal objective of this section.

Before entering into conformality, let us briefly review the alternative. The experimental data give couplings at the Z pole of [26] $\alpha_3 = 0.118 \pm 0.003$, $\alpha_2 = 0.0338$, $\alpha_1 = \frac{5}{3}\alpha'_Y = 0.0169$ (where the errors on $\alpha_{1,2}$ are less than 1%) and $\sin^2\theta = \alpha'_Y/(\alpha_2 + \alpha'_Y) = 0.231$ with an error less than 0.001. Note that α_2/α_1 is very nearly two; this will be used later. The RGE for the supersymmetric grand unification [27,28] are

$$\frac{1}{\alpha_i(M_G)} = \frac{1}{\alpha_i(M_Z)} - \frac{b_i}{2\pi} \ln\left(\frac{M_G}{M_Z}\right) \quad (19)$$

Using the MSSM values $b_i = (6\frac{3}{5}, 1, -3)$ and substituting $\alpha_{2,3}$ at $M_Z = 91.187\text{GeV}$ gives $M_G = 2.4 \times 10^{16}\text{GeV}$ and $\alpha_{2,3}(M_G)^{-1} = 24.305$. Using Eq(19) with $i = 1$ now predicts $\alpha_1(M_Z) = 59.172$ and hence $\sin^2\theta = 0.231$; this is impressive agreement with experiment and is sometimes presented as the accurate meeting of three straight lines on a $\alpha_i^{-1}(\mu)$ vs. $\ln\mu$ plot [18,19].

As we have seen above, the relationship of the Type IIB superstring to conformal gauge theory in $d = 4$ gives rise to an interesting class of gauge theories. Choosing the simplest compactification [10] on $AdS_5 \times S_5$ gives rise to an $\mathcal{N} = 4$ SU(N) gauge theory which is known to be conformal due to the extended global supersymmetry and non-renormalization theorems. All of the RGE β -functions for this $\mathcal{N} = 4$ case are vanishing in perturbation theory. It is possible to break the $\mathcal{N} = 4$ to $\mathcal{N} = 2, 1, 0$ by replacing S_5 by an orbifold S_5/Γ where Γ is a discrete group with $\Gamma \subset SU(2), \subset SU(3), \not\subset SU(3)$ respectively.

In building a conformal gauge theory model [11–13], the steps are: (1) Choose the discrete group Γ ; (2) Embed $\Gamma \subset SU(4)$; (3) Choose the N of $SU(N)$; and (4) Embed the Standard Model $SU(3) \times SU(2) \times U(1)$ in the resultant gauge group $\otimes SU(N)^p$ (quiver node identification). Here we shall look only at abelian $\Gamma = Z_p$ and define $\alpha = \exp(2\pi i/p)$. It is expected from the string-field duality that the resultant field theory is conformal in the $N \rightarrow \infty$ limit, and will have a fixed manifold, or at least a fixed point, for N finite.

Before focusing on $\mathcal{N} = 0$ non-supersymmetric cases, let us first examine an $\mathcal{N} = 1$ model first put forward in [9]. The choice is $\Gamma = Z_3$ and the **4** of $SU(4)$ is $\mathbf{4} = (1, \alpha, \alpha, \alpha^2)$. Choosing $N=3$, this leads to the three chiral families under $SU(3)^3$ trinification [29]

$$(3, \bar{3}, 1) + (1, 3, \bar{3}) + (\bar{3}, 1, 3) \quad (20)$$

In this model it is interesting that the number of families arises as $4-1=3$, the difference between the **4** of $SU(4)$ and $\mathcal{N} = 1$, the number of unbroken supersymmetries. However this model has no gauge coupling unification; also, keeping $\mathcal{N} = 1$ supersymmetry is against the spirit of the conformality approach. We now present three examples, Models A, B, and C, which accommodate three chiral families, break all supersymmetries ($\mathcal{N} = 0$), and possess gauge coupling unification, including the correct value of the electroweak mixing angle.

Model A. Choose $\Gamma = Z_7$, embed the **4** of $SU(4)$ as $(\alpha^2, \alpha^2, \alpha^{-3}, \alpha^{-1})$, and choose $N=3$ to aim at a trinification $SU(3)_C \times SU(3)_W \times SU(3)_H$.

The seven nodes of the quiver diagram will be identified as C-H-W-H-H-H-W.

The behavior of the **4** of $SU(4)$ implies that the bifundamentals of chiral fermions are in the representations

$$\sum_{j=1}^7 [2(N_j, \bar{N}_{j+2}) + (N_j, \bar{N}_{j-3}) + (N_j, \bar{N}_{j-1})] \quad (21)$$

Embedding the C, W and H $SU(3)$ gauge groups as indicated by the quiver mode identifications then gives the seven quartets of irreducible representations:

$$\begin{aligned}
& [3(3, \bar{3}, 1) + (3, 1, \bar{3})]_1 + \\
& + [3(1, 1, 1 + 8) + (\bar{3}, 1, 3)]_2 + \\
& + [3(1, 3, \bar{3}) + (1, 1 + 8, 1)]_3 + \\
& + [(2(1, 1, 1 + 8) + (1, \bar{3}, 3) + (\bar{3}, 1, 3)]_4 + \\
& + [2(1, 1, 1 + 8) + 2(1, \bar{3}, 3)]_5 + \\
& + [2(\bar{3}, 1, 3) + (1, 1, 1 + 8) + (1, \bar{3}, 3)]_6 + \\
& + [4(1, 3, \bar{3})]_7.
\end{aligned} \tag{22}$$

Combining terms gives, aside from (real) adjoints and overall singlets

$$3(3, \bar{3}, 1) + 4(\bar{3}, 1, 3) + (3, 1, \bar{3}) + 7(1, 3, \bar{3}) + 4(1, \bar{3}, 3) \tag{23}$$

Cancelling the real parts (which acquire Dirac masses at the conformal symmetry breaking scale) leaves under trification $SU(3)_C \times SU(3)_W \times SU(3)_H$

$$3[(3, \bar{3}, 1) + (1, 3, \bar{3}) + (\bar{3}, 1, 3)] \tag{24}$$

which are the desired three chiral families.

Given the embedding of Γ in $SU(4)$ it follows that the 6 of $SU(4)$ transforms as $(\alpha^4, \alpha, \alpha, \alpha^{-1}, \alpha^{-1}, \alpha^{-4})$. The complex scalars therefore transform as

$$\sum_{j=1}^7 [(N_j, \bar{N}_{j\pm 4}) + 2(N_j, \bar{N}_{j\pm 1})] \tag{25}$$

These bifundamentals can by their VEVs break the symmetry $SU(3)^7 = SU(3)_C \times SU(3)_W^2 \times SU(3)_H^4$ down to the appropriate diagonal subgroup $SU(3)_C \times SU(3)_W \times SU(3)_H$.

Now to the final aspect of Model A which is its motivation, the gauge coupling unification. The embedding in $SU(3)^7$ of $SU(3)_C \times SU(3)_W^2 \times SU(3)_H^4$ means that the couplings $\alpha_1, \alpha_2, \alpha_3$ are in the ratio $\alpha_1/\alpha_2/\alpha_3 = 1/2/4$. Using the phenomenological data given at the beginning, this implies that $\sin^2\theta = 0.231$.

On the other hand, the QCD coupling is $\alpha_3 = 0.0676$ which is too low unless the conformal scale is at least 10TeV. We prefer a scale ~ 1 TeV for conformal breaking where α_3 is nearer to 0.10. This motivates our Models B and C below which have larger α_3 but are otherwise more complicated.

Model B. Choose $\Gamma = Z_{10}$ and embed $Z_{10} \subset SU(4)$ such that $4 = (\alpha^4, \alpha^4, \alpha^{-3}, \alpha^{-5})$. The chiral fermions are therefore

$$\sum_{j=1}^{10} [2(N_j, \bar{N}_{j+4}) + (N_j, \bar{N}_{j-3}) + (N_j, \bar{N}_{j-5})] \quad (26)$$

To attain trinification we identify the quiver nodes as C-H-H-H-W-W-H-W-H-H and then the chiral fermions are in the ten quartets of irreducible representations

$$\begin{aligned} & [4(3, \bar{3}, 1)]_1 + \\ & + [2(1, \bar{3}, 3) + (1, 1, 1+8)]_2 + \\ & + [2(1, 1, 1+8) + (1, \bar{3}, 3)]_3 + \\ & + [2(1, \bar{3}, 3) + (\bar{3}, 1, 3) + (1, 1, 1+8)]_4 + \\ & + [4(1, 3, \bar{3})]_5 + \\ & + [3(1, 3, \bar{3}) + (\bar{3}, 3, 1)]_6 + \\ & + [2(\bar{3}, 1, 3) + (1, 1, 1+8)]_7 + \\ & + [3(1, 3, \bar{3}) + (1, 1+8, 1)]_8 + \\ & + [3(1, 1, 1+8) + (1, \bar{3}, 3)]_9 + \\ & + [3(1, 1, 1+8) + (1, \bar{3}, 3)]_{10} \end{aligned} \quad (27)$$

Removing the (real) octets and singlets leaves

$$4(3, \bar{3}, 1) + (\bar{3}, 3, 1) + 3(\bar{3}, 1, 3) + 10(1, 3, \bar{3}) + 7(1, \bar{3}, 3) \quad (28)$$

so that the chiral (complex) part is again

$$3[(3, \bar{3}, 1) + (1, 3, \bar{3}) + (\bar{3}, 1, 3)] \quad (29)$$

which are three chiral families.

The 6 of $SU(4)$ transforms under $\Gamma = Z_{10}$ as $6 = (\alpha^8, \alpha, \alpha, \alpha^{-1}, \alpha^{-1}, \alpha^{-8})$ and so the complex scalars are

$$\sum_{j=1}^{10} [(n_j, \bar{N}_{j\pm 8}) + 2(N_j, \bar{N}_{j\pm 1})] \quad (30)$$

With the given quiver node identification VEVs for these scalars can break $SU(3)^{10} = SU(3)_C \times SU(3)_W^3 \times SU(3)_H^6$ to the diagonal subgroup $SU(3)_C \times SU(3)_W \times SU(3)_H$.

The couplings $\alpha_1, \alpha_2, \alpha_3$ are in the ratio $\alpha_1/\alpha_2/\alpha_3 = 1/2/6$ corresponding to $\sin^2\theta = 0.231$ and $\alpha_3 = 0.101$. This is within the range of a TeV conformal breaking scale. Nevertheless, it is numerically irresistible to notice that the Z-pole values satisfy $\alpha_1/\alpha_2/\alpha_3 = 1/2/7$ which leads naturally to Model C.

Model C. Choose $\Gamma = Z_{23}$ and embed in $SU(4)$ by $4 = (\alpha^6, \alpha^6, \alpha^{-5}, \alpha^{-7})$. Given this embedding the quiver nodes can be chosen as C-C-X-X-X-H-H-W-H-X-X-X-X-X-X-W-H-H-W-X-X-X where the thirteen X's denote any distribution of four W's and nine H's that allows breaking by the complex scalars cited below. The quiver is arranged such that according to the rule of $(3_C - \bar{3}_W)$ minus $(3_W - \bar{3}_C)$ there are three chiral families. [The model in [13] did not follow this rule and has two families.] Note that because of anomaly cancellation and the occurrence of only bifundamentals the remainder of trinification is automatic and need not be checked in every case.

The chiral families are as in Models A and B.

The 6 of $SU(4)$ transforms as $(\alpha^{12}, \alpha, \alpha, \alpha^{-1}, \alpha^{-1}, \alpha^{-12})$. This implies complex scalars whose VEVs can break $SU(3)^{23} = SU(3)_C^2 \times SU(3)_W^7 \times SU(3)_H^{14}$ to $SU(3)_C \times SU(3)_W \times SU(3)_H$ with a suitable distribution of W and H nodes on the quiver.

With this choice of diagonal subgroups the couplings are in the ratio $\alpha_1/\alpha_2/\alpha_3 = 1/2/7$ corresponding to $\sin^2\theta = 0.231$ and $\alpha_3 = 0.118$ which coincide with the Z-pole values.

In this section, we have given three examples of building conformal models from abelian Γ with acceptable values of the couplings at the conformal scale, assuming that the $SU(3)$ gauge couplings are all equal at the conformal scale. Model A is the simplest but its α_3 is

too small unless the conformal scale is taken up to at least 10TeV. Models B and C can accommodate a lower conformal scale but are more complicated.

There are two features of conformal models which bear repetition:

(1) Bifundamentals prohibit representations like (8,2) or (3,3) in the Standard Model consistent with Nature.

(2) Charge quantization is incorporated since the abelian $U(1)_Y$ group has a positive-definite β -function and cannot be conformal until it is embedded in a non-abelian group.

There are three questions which merit further investigation:

(1) The first question bears on whether there is a fixed manifold (line, plane,...) with respect to the renormalization group or only a fixed point which is, in any case, sufficient to apply our conformality constraints. In perturbation theory, do the β -functions vanish?

(2) Are the additional particles necessary to render the Standard Model conformal consistent with the stringent constraints imposed by the precision electroweak data?

(3) Coefficients of dimension-4 operators are prescribed by group theory and all dimensionless properties such as quark and lepton mass ratios and mixing angles are calculable. Do these work and, if not, can one refine the model-building to obtain a best fit?

V. DISCUSSION

String theory has existed for over thirty years and its connection with the real world (at least for ten/eleven dimensional versions) is unknown despite a multitude of attempts. The AdS/CFT correspondence offers, in our opinion, the most promising approach presently available to relate string theory to observable physics.

The use of the AdS/CFT correspondence involves the step of, in the first order, dropping the gravitational interaction. In any foreseeable high-energy experiment gravity will be negligible so the approximation is reasonable. On the other hand, if the particles predicted by the conformality approach discussed in this article were to be discovered in the next (TeV) energy regime, it would provide support for the string approach, including as a theory of quantum gravity.

Whether or not string theory is the correct unifying theory with gravity, it does provide through the AdS/CFT correspondence very promising ideas of how to write conformal theories in four space-time dimensions with particular semi-simple gauge groups, chiral fermions and complex scalars and one of these theories could be the correct direction to proceed.

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